NET 2025 Power Round

Advanced Division

April 2025

Instructions

This test consists of six questions. While you are free to attempt all six questions, we will only grade your four best-performing questions, regardless of how well you do on the other two. A question's point value is *not* informative of its difficulty; although questions have different point values, each question is weighted independently of its point value in your final cumulative score. After normalizing point-values of each question to the same weight, your cumulative score will be calculated as the sum of the scores of your four best-performing questions. You are encouraged to work together on these questions. Answer each question as clearly and succinctly as possible. You may write on a blank sheet of paper where you *clearly indicate* where your answer to each part is. If you are unsure of your answer, take your best guess: there is no penalty for incorrect answers. If you find yourself stuck on a question, skip it and return to it at the end if necessary. You will have two hours (120 minutes) to complete the exam. Remember, we do *not* share your answers or scores with Northwestern admissions, nor do we keep them for ourselves. You are not expected to know how to answer each question on the exam; rather, this test is designed to assess your economic and formal reasoning skills. Have fun, and good luck!

Problem 1: Portfolio Valuation, Futures, and Options Arbitrage (25 points)

Financial markets consist of various assets, including stocks, bonds, futures, and options, which traders use for speculation, hedging, and arbitrage. This problem explores portfolio valuation, the no-arbitrage principle, and the pricing behavior of options and futures contracts.

Part A *(4 points) Portfolio Valuation and Present Value*

(1) *(2 points)* Consider a market consisting of a stock S with price $S(t)$ at time t and bonds/money market instruments that grow at a risk-free interest rate r, compounded continuously. Suppose you construct a portfolio consisting of 5 shares of stock and \$10 worth of bonds at time zero. Derive an expression for the value of this portfolio in three years under continuous compounding.

(2) *(2 points)* The present value of an asset is its current worth, meaning it is the amount of money today when invested at the interest rate and compounded over time required to equal the value of an asset at some future time when. Compute the present value of the portfolio you derived in part (1).

Part B *(4 points) Futures Pricing and Arbitrage:* A futures contract is a standardized financial agreement between two parties (made at some time $t < T$) to buy or sell an asset at a pre-determined price F on a specified future date T. The fair value of a futures contract is given by:

$$F = S(t)e^{r(T-t)} \tag{1}$$

(1) *(2 points)* Calculate the theoretical fair value of a futures contract on stock S one year before expiration, given that the current stock price is \$105 and the risk-free rate is 5% per year. Use the approximation $e^x \approx 1 + x$.

(2) *(2 points)* The no-arbitrage principle states that in an efficient market, there should be no opportunity to make a risk-free profit without investing any capital (i.e. it should not be possible to have a portfolio with worth = \$0 at time $t < T$ and a portfolio with zero probability of being worth negative and positive probability of being positive at time T). Suppose the observed futures price is \$112. Does an arbitrage opportunity exist? If so, describe a risk-free strategy that exploits this mispricing, assuming you can borrow or invest money at the risk-free rate, buy or sell the underlying stock, and buy or sell a futures contract.

Part C *(6 points) Option Pricing and Intrinsic Value*

(1) *(2 points)* Unlike futures, an option contract gives the right, but not the obligation, to buy (call option) or sell (put option) a stock $S(t)$ at a predetermined strike price K at a specified expiration time T. The intrinsic value of an option is the immediate profit an option holder would realize if they exercised the option at time T. Derive expressions for the intrinsic value of both call and put options at expiration. Additionally, draw a payoff diagram for both a call and put option with strike price K as a function of $S(T)$.

(2) *(4 points)* Consider a binomial model where stock S moves up by a factor u with probability p or down by a factor d with probability 1−p. A call option's price today is the present value of the weighted average of possible intrinsic values (based on the different possible values of $S(T)$ at expiration, where probability weights are determined by the risk-neutral probability q, satisfying:

$$r = qu + (1-q)d \tag{2}$$

(This represents the probability of the stock going up such that the expected return of the stock is equal to the risk-free rate r). Given: $S(0) = 100$, $K = 100$, $T = 2$ periods, $u = 3\%$, d=-2%, $r = 1\%$ per period (assume for this problem, that the interest rate compounds every period, so no continuous compounding), calculate the value of the European call option today.

Part D *(2 points) Early Exercise of American Options* The previous part explored European options, which only allow the holder to exercise at the expiration time T. The American option, on the other hand, allows the holder to exercise at any time $t \leq T$. Assuming no dividend payments, is it ever optimal to exercise an American call option early? If so, why, and under what conditions? (Hint: consider separately cases where $S(t) \geq K$ and $S(t) < K$ after exercising.)

Part E *(4 points) Put-Call Parity and Arbitrage*

(1) *(2 points)* Put-call parity is a fundamental principle in options pricing that establishes a relationship between the price of a call option, a put option, the underlying asset, and a risk-free bond. It ensures that arbitrage opportunities do not exist between call and put options in efficient markets. Derive this equation by solving for the difference between a call and put option each with strike price K at time t $(c(t) - p(t))$ in terms of $S(t)$, K, and r. (Hint: Find the portfolio of the stock and bonds at time T with the same value as c(T) - p(T). What does the no arbitrage assumption say about this portfolio's value at time t?)

(2) *(2 points)* Suppose at time 0, a European call and put option both with strike price \$101.50 and expiration in 6 months are priced as follows: Call price: \$6.50, Put price: \$5.50, Stock price: \$102, Interest rate: 3% per year (compounded bi-annually). Check whether put-call parity holds. If not, describe a risk-free arbitrage strategy involving the underlying asset, investing/borrowing money at the interest rate, exactly one call option, and exactly one put option, and compute the maximum arbitrage profit.

Part F *(5 points) Behavior of Option Prices*

(1) *(2 points)* How do the prices of a call and put option change as the strike price increases? Draw a graph of call and put option prices as a function of strike price K, while fixing the spot price S for some K.

(2) *(2 points)* How do the prices of a call and put option change as the underlying stock price increases? Additionally, how do the prices of a call and put option (at some time $t < T$) compare to their intrinsic value (their worth at time T)? Draw a graph of call and put option prices as a function of the underlying stock price S, while fixing strike price K or some spot price S. Compare prices at times $t < T$ and T, and explain the differences.

(3) *(1 point)* The time value of an option is the difference between its current price and intrinsic value. For a fixed strike price K, at what level of the underlying stock price S does the time value of an option reach its maximum? Explain intuitively why this occurs.

Problem 2: The Tariff-ic Game (25 points)

In this problem, we explore a game theoretic approach to the ramifications of tariffs.

Preliminaries Consider two identical countries. Each country has a government, a firm and consumers who buys from the two firms. (Assume the governments, firms and consumers are completely identical in each country, and that the firms produce identical goods.) In this game, governments first choose to levy a tariff, denoted t_i. Then, firms choose how much they want to produce for domestic consumption and exports, denoted h_i and e_i, respectively. Country i's firm has a market demand curve given by $P_i = a - h_i - e_j$, where h_i denotes the quantity produced for home consumption and e_j denotes the quantity exported to country i from country j.

Part A *(2 points)* Suppose both firms have a constant marginal cost, c. What is the total cost function for firm i? Firm j?

Part B *(1 point)* What is the profit function (denoted π_i) for each firm?

Part C *(2 points)* Each government seeks to maximize its total welfare function, which can be expressed as the sum of consumer surplus, producer surplus, and revenue collected from the tariff. What is the welfare function for government i?

Part D *(3 points total)* Suppose government i levies a fixed tariff of t_i^* and government j levies a fixed tariff of t_j^*. Further suppose firm j fixes quantities h_j^* and e_j^*. Compute the profit maximizing quantities, h_i^* and e_i^*, chosen by firm i as a function of t_i^*, t_j^*, h_j^*, e_j^* and constants. *(Hint: consider breaking this problem into two separate maximization problems.)*

Part E *(1 point)* Now compute the profit maximizing quantities for firm j as a function of t_i^*, t_j^*, h_i^*, e_i^* and constants. *(Hint: the firms are symmetric!)*

Part F *(2 points)* Solve the resulting system of equations for h_i^*, e_i^*, h_j^*, and e_j^* as a function of the fixed tariffs and constants.

Part G *(1 point)* Rewrite the welfare function for government i in terms of tariffs, t_i and t_j, and constants.

Part H *(2 points)* Assume government j fixes a tariff of t_j^*. Solve for the welfare-maximizing tariff t_i^* chosen by government i.

Part I *(1 point)* What is the welfare-maximizing tariff t_j^* chosen by government j assuming government i fixes a tariff of t_i^*? *(Hint: the governments are also symmetric!)*

Part J *(1 point)* Using the tariffs you found in Part I, solve for the quantities produced by firms i and j in terms of constants.

Part K *(2 points)* Show that having zero tariffs maximizes total welfare across both countries.

Part 6 *(7 points total)* In the famous Cournot competition model, two firms in the same market simultaneously choose a quantity q to produce. The demand curve for both firms is given by $P = a - q_i - q_j$, where q_i and q_j denote the quantities chosen by each firm.

(1) *(2 points)* Solve for the profit maximizing quantity chosen by each firm as a function of the quantity produced by the other firm.

(2) *(1 point)* Solve the system of equations for the profit maximizing quantity chosen by each firm.

(3) *(4 points)* Compare the outcome of the Cournot game to the outcome of the tariffs game. How does the quantity produced by each firm differ? What about market price? In which game are consumers better off? Producers?

Problem 3: Weird Guy at Birthday Party (25 points)

In this problem, we consider ways to divide goods among multiple individuals so that everyone is satisfied. Various notions of what this means in particular will be elaborated on and explored.

Preliminaries We begin by defining two potential criteria for fair division: *proportionality* and *envy-freeness*. Suppose for $i \in \{1, 2, ..., n\}$, each agent i receives a share worth X_i, where the entire cake is worth $1 = \sum_{i=1}^{n} X_i$.

(1) We say the division is *proportional* if for all $i \in \{1, 2, ..., n\}$, $X_i \geq 1/n$.
I.e. each agent receives at least $1/n$ of the whole cake.

(2) We say the division is *envy-free* if for all i, $j \in \{1, 2, ..., n\}$, $X_i \succeq_i X_j$.
I.e. each agent weakly prefers their share to any other share.

Part A *(2 points total)* Suppose there are 2 agents, Azir and Bard, attempting to split a cake. Define the cake as the interval $[0, 1]$, where each agent values their piece based on the length of the line segment they receive. We can make a proportional and envy-free division by using the following process. First, Azir chooses a cut $k \in [0, 1]$ where he prefers both sides equally. Then, Bard chooses either the left $[0, k]$ or right piece $(k, 1]$.

(1) *(1 point)* Explain why this division is proportional.

(2) *(1 point)* Explain why this division is envy-free.

Part B *(1 point)* An additional criteria we may consider is *symmetry*. We say the division is *symmetric* if every agent is guaranteed the same value no matter the permutation of roles the agents receive. Explain why the division from Part A is not symmetric.

Part C *(2 points)* Consider the following process. Azir and Bard each make a mark where they believe the halfway point is at a and b respectively. Then a cut is made at $(a + b)/2$. Explain how the process should continue to ensure a symmetric, proportional, envy-free division.

Part D *(3 points)* Suppose we have the cake from $[0, 1]$ as defined in Part A. However, there is a 3rd person, Caitlyn, who wishes to eat the cake too. We want to define a proportional and envy-free division. We proceed in the following way:

(1) Azir divides the cake into thirds, creating pieces X, Y, Z.

(2) Suppose Bard believes X is the biggest piece. Then, Bard trims X to be the same as the 2nd largest piece. Let X^L be the larger trimmed piece of X and let X^s be smaller trimming of X.

(3) Set aside X^s to be divided later. We proceed with dividing X^L, Y, and Z.

(4) Caitlyn chooses between X^L, Y, and Z.

(5) If Caitlyn did not take X^L, Bard must take X^L. Otherwise, Bard chooses from the remaining two pieces.

(6) Azir takes the last piece.

Suppose after Step 1, Bard believes the two largest pieces are the same size. Describe how we can continue the process to ensure the division is proportional and envy-free. Justify why the division meets those criteria.

Part E *(6 points total)* Suppose after Step 1, Bard believes there is one piece that is strictly larger than the others. Then, we proceed with Step 2-6. We have divided up X^L, Y, and Z, but the trimming X^s still remains to be divided.

(1) *(3 points)* Describe how we can continue the process to ensure the division is proportional and envy-free. (Hint: let Bard or Caitlyn, whoever did not choose X^L, make cut(s). Then, come up with an ordering of "choosers".)

(2) *(3 points)* Justify that your division in Part E is envy-free.

Part F *(3 points total)* Suppose Azir, Bard, and Caitlyn receive another cake they want to divide. This time, the cake is 2-dimensional. First, each of the three makes a vertical mark so that the piece to the left of the mark is worth 1/3. Suppose without loss of generality that Azir made the leftmost mark. Then, Azir receives the left piece.

(1) *(1 point)* Suppose we divide the right piece by using the division from Part A. Explain why Bard and Caitlyn do not envy anyone.

(2) *(2 points)* Explain why the process as a whole is not envy-free.

Part G *(8 points total)* Instead of splitting the remaining right piece using the division from Part A, we will proceed as follows. Azir will rotate his knife over the cake for each angle $\theta \in [0, \pi]$ such that he believes each side is exactly 1/2. (It may help to imagine the remaining piece as a circle, although the specific shape is unimportant to the problem).

(1) *(1 point)* Let B be Bard's valuation of the remaining piece. Consider $X_b(\theta)$, Bard's valuation of the piece left of Azir's knife at angle θ. Express $X_b(\pi)$ in terms of $X_b(0)$, B.

(2) *(3 points)* Explain why there is guaranteed to be an angle θ for which $X_b(\theta) = \frac{1}{2}B$.

(3) *(2 points)* Describe how the process should continue to guarantee it is envy-free.

(4) *(2 points)* Justify that the process is envy-free.

Problem 4: When YAKs Get Smarter (25 Points)

In this problem we consider how the growth rate of technologies usefulness can impact the returns to labor and capital by building on a basic growth model. This is a model where learning by doing is possible.

Preliminaries This question concerns a country, called Yakland, with Cobb Douglas growth and interchangeable human capital and physical capital. In this question, consider upper case letters to represent total values while lower case letters represent per capita values. Thus Y represents output while y represents output per capita or $y = \frac{Y}{L}$, where L is the population, and K represents capital while k is capital per capita.

Part A *(1 Point)* In this economy output is the product of capital, K, and the productivity of capital, denoted A. Write this relationship as an equation.

Part B *(1 Point)* Solve this equation for the per capita values of output. Do not include L in the equation.

Part C *(2 Points)* Assume that capital depreciates at the rate δ and there is no growth in population. Consumer save a portion of their income, denoted s. These savings are invested into physical capital. Find an equation for the next period's capital, denoted K_{t+1} as a function of present capital K_t, depreciation, the savings rate and the current period's output denoted Y_t.

Part D *(1 Point)* Find the growth equation of capital per capita. Remember there is no growth in population.

Part E *(1 Point)* A country is at a steady state if there is no change capital from one period to another. Find the condition required for Yakland to be at a steady state. Remember that output is function of capital as well.

Part F *(2 Points)* Suppose we have a country with N firms. We denote the production of each firm

$$y_i = \bar{A} k_j^{\alpha} L_j^{1-\alpha}$$

where

$$\bar{A} = A_0 (\sum_{i=1}^{N} k_j)^{\tau}$$

Describe how a firm's decision to increase capital effects $\bar{A}$ and what that means for the utilization of capital.

Part G *(1 Points)* Assume the firms are symmetric and normalize L_j to 1. Represent the countries total capital, K as a function of k_j and N. Represent the country's total output, Y as a function of y_j and N.

Part H *(2 Points)* Use the prior question to find $\bar{A}$ in terms of non-firm (without indices) variables.

Part I *(1 Point)* Now represent the country's total output only as a function of non-firm variables.

Part J *(2 Points)* Recall that $K_{t+1} = sY - \delta K$. Use that the law of capital growth is the same as found in part C (not in part D). Under what conditions does this function operate equivalently to the model discussed in Part E? Show the most simplified equivalence.

Part K *(3 Points)* Under what condition will the steady state converge to the same equation as part E? Under what condition will this economy experience explosive growth? Demonstrate how the economy will act when above or below the steady state.

Part L *(2 Points)* Find an equation for the steady state of capital, denoted K^* under the steady state condition.

Part M *(2 Points each)* On 3 separate graphs consider an economy in the steady state. Demonstrate how the level of capital changes before and after each shock (independently).

- *(2 Points)* An unexpected and permanent increase in the savings rate
- *(2 Points)* An unexpected inflow of foreign aid in the form of capital
- *(2 Points)* The unexpected exit of a significant number of firms from the market to other countries

Problem 5: Pricing Perfect Purple Pens (24 Points)

In this problem, we analyze how firms set prices when dealing with homogeneous products in limited and infinite periods.

Preliminaries Suppose there is a market in Greendale where two firms (Firm A and Firm B) produce purple pens. In this market, assume that each firm simultaneously sets a price for purple pens. Consumers then decide how many purple pens to buy and which firm to buy from.

Part A Suppose that Firm A and Firm B produce homogenous (i.e. identical) products. In this market, if one price is less than the other price, consumers only buy purple pens from the lower priced firm. If both prices are the same, each firm sells half of the total output demanded at that price. Total demand at the purchased price is determined by the demand curve listed below. Assume α and β are positive constants and each firm has a unit cost of production of c:

$$P = \alpha - \beta * Q$$

(2 Points) In this market, write the respective profit functions for Firm A and Firm B as functions of α and β. (Hint: the profit functions are discontinuous.)

Part B *(2 Points)* Given the above profit functions, what is the unique Nash equilibrium price for Firm A and Firm B. (Hint: the Nash equilibrium price for Firm A and Firm B will be the price such that neither firm has the incentive to switch prices after knowing the price the other firm has set.)

Part C *(1 Point)* Given the Nash equilibrium prices, what are the profits for Firm A and Firm B?

Part D Now suppose that Firm A and Firm B **don't** produce homogenous products (i.e. their products are differentiated). Suppose that the two firms sell products whose demand curves are listed below. Assume that α and β are positive constants and that each firm has **zero costs of production**.

$$\text{Firm A: } q_1 = \alpha - \beta * p_1 + x(p_2 - p_1)$$
$$\text{Firm B: } q_2 = \alpha - \beta * p_2 + x(p_1 - p_2)$$

(2 Points) In this market, write the respective profit functions for Firm A and Firm B as functions of α, β, p_1, p_2, and x.

Part E *(2 Points)* Given the above profit functions, what is the unique Nash equilibrium price for Firm A (p_1) and Firm B (p_2) in terms of x. (Hint: set the first derivatives of the profit functions equal to zero).

Part F *(1 Point)* Given the Nash equilibrium prices, what are the profits for Firm A and Firm B?

Part G *(1 Point)* Do increases in x cause prices to fall or rise? Provide an economic explanation for this effect.

Part H *(2 Points)* Compare the Nash equilibrium profits and prices for both firms in Part A and B and provide an economic explanation for the difference.

Part I *(3 Points)* Consider the market given in Part A and assume the same conditions (both firms simultaneously announce a price and if one of the prices is lower than the other, the firm charging the lower price sells the amount demanded at that price, and the firm charging the higher price sells nothing). Now assume that the firms play the following stage game an infinite number of times, where both firms simultaneously announce a price at the beginning of each period. Assume that both firms have a discount rate of $0 < \delta < 1$ in each period.

(5 Points) Considering price can be any value between c and α, what is the minimum discount rate δ in order for there to exist a Nash equilibrium where both firms charge a price of $p^* = (\alpha - c)/2\beta$ in each period?

Part K *(5 Points)* Now assume price can be any value between $(\alpha + c)/2$ and α. Calculate a formula for $\delta(p^*)$, which is a function for the minimum discount rate in order for there to exist a Nash equilibrium where both firms charge a price of p^*.

Problem 6: Two-Period Extraction (25 points)

In this problem, we build out an economic model of natural resource extraction across two time periods.

Preliminaries The utilization of nonrenewable natural resources is a fundamentally dynamic problem: it is crucial to optimize not just how much of a resource is used but also when that use occurs. This problem uses the example of mineral extraction, a live issue in economics and geopolitics today, as window dressing to explore optimal resource consumption.

Part A *(13 points total)* Suppose a firm owns a fixed stock (x) of minerals across its mines. The firm seeks to maximize profits across two periods (t_0 and t_1) by selling some amount of minerals (y_0) in the first and the rest in the second. The firm faces constant marginal costs (c) and discount factor ρ. [Note: ρ is derived from the discount rate of the firm and serves to define t_1 profits in present value terms.]

The firm makes profits from mineral extraction during both periods in terms of y_0, first period price (p_0), and second period price (p_1) can therefore be expressed as:

$$\pi(y_0, p_0, p_1) = (p_0 - c)y_0 - \rho(p_1 - c)(x - y_0)$$

(1) *(3 points)* Assuming a positive quantity of minerals is sold during each period, the firm will set quantities such that an equilibrium between t_0 and t_1 is reached. This is known as the no-intertemporal arbitrage condition. Derive the first order condition with respect to y_0 of the above equation to find that relationship. Then, provide some brief intuition as to what the condition means for the firm. [Hint: you should find some relationship between p_0 and p_1.]

(2) *(5 points)* Given stock size, costs, and mineral demands in the two periods as well as ρ, we are able to predict firm behavior in equilibrium. Solve for mineral prices (p_0 and p_1) and extraction levels (y_0 and y_1) to the nearest hundredth in both periods using the following parameters. How do prices and quantities differ between the periods? [Hint: remember that the entire resource stock is extracted.]

$$y_0 = 16 - p_0$$
$$y_1 = 12 - p_1$$
$$x_0 = 8$$
$$\rho = 0.84$$
$$c = 1.5$$

(3) *(3 points)* We might worry about the negative environmental and social impacts of mineral extraction. Suppose a consortium of major companies announce their plans to reduce mineral purchases in period 1, reducing demand to the equation below. All other parameters remain unchanged. Re-calculate the quantities (p_0, p_1, y_0, y_1) from the last task. What happens to extraction and prices in each period?

$$y_1 = 10 - p_1$$

(4) *(2 points)* German economist Hans-Werner Sinn identifies this phenomenon as the "Green Paradox," where well-intentioned environmental commitments for the future can increase extraction—and its associated harms—today. Explain why setting green demand-side restrictions for the future might still be worthwhile. Then, describe a sustainable policy that Sinn might recommend instead.

Part B *(12 points total)* In reality, it is unlikely that costs remain static with respect to resource stock and extraction level. We might instead expect prices to rise along both dimensions, with firms utilizing the most accessible resources first.

$$c(x, y) = C(\sigma + x)^{-\alpha} y^{1+\beta}$$

(1) *(4 points)* The above equation represents one way that mineral stocks and extraction rates could influence costs. What do the alpha (α), beta (β), and sigma (σ) parameters indicate about the firm's costs? Given the full-extraction assumption, what must be true about sigma? [Hint: consider cases where the parameters are zero and positive.]

(2) *(2 points)* We can use this advanced cost equation to inform a new equilibrium condition. We continue to adhere to the assumptions that extraction is positive in both periods and that all available minerals are extracted. Use the above relationship to construct a profit equation, similar to the one stated in Part A, in terms of y_0 and x_0. Represent period costs as a function of x and y as shown above.

(3) *(6 points)* To find a mineral firm's profits with variable costs, take the first order condition of the equation you found above with respect to y_0. Arrange the resulting equation such that the discounted terms fall on the same side. Then, briefly describe the real-world significance of each term (i.e., how each term explains changes in profit across some dimension.) [Hint: Take partial derivatives of the cost function. Use your knowledge of the relationship between x and y to determine $\mathrm{d}x_1/\mathrm{d}y_0$ and $\mathrm{d}y_1/\mathrm{d}y_0$.]