NET 2025 Power Round: Solutions

Advanced Division

April 2025

Instructions

This test consists of six questions. While you are free to attempt all six questions, we will only grade your four best-performing questions, regardless of how well you do on the other two. A question's point value is *not* informative of its difficulty; although questions have different point values, each question is weighted independently of its point value in your final cumulative score. After normalizing point-values of each question to the same weight, your cumulative score will be calculated as the sum of the scores of your four best-performing questions. You are encouraged to work together on these questions. Answer each question as clearly and succinctly as possible. You may write on a blank sheet of paper where you *clearly indicate* where your answer to each part is. If you are unsure of your answer, take your best guess: there is no penalty for incorrect answers. If you find yourself stuck on a question, skip it and return to it at the end if necessary. You will have two hours (120 minutes) to complete the exam. Remember, we do *not* share your answers or scores with Northwestern admissions, nor do we keep them for ourselves. You are not expected to know how to answer each question on the exam; rather, this test is designed to assess your economic and formal reasoning skills. Have fun, and good luck!

Problem 1: Portfolio Valuation, Futures, and Options Arbitrage (25 points)

Financial markets consist of various assets, including stocks, bonds, futures, and options, which traders use for speculation, hedging, and arbitrage. This problem explores portfolio valuation, the no-arbitrage principle, and the pricing behavior of options and futures contracts.

Part A *(4 points) Portfolio Valuation and Present Value*

(1) *(2 points)* Consider a market consisting of a stock S with price $S(t)$ at time t and bonds/money market instruments that grow at a risk-free interest rate r, compounded continuously. Suppose you construct a portfolio consisting of 5 shares of stock and \$10 worth of bonds at time zero. Derive an expression for the value of this portfolio in three years under continuous compounding.

Part B *(4 points) Futures Pricing and Arbitrage:* A futures contract is a standardized financial agreement between two parties (made at some time $t < T$) to buy or sell an asset at a predetermined price F on a specified future date T. The fair value of a futures contract is given by:

$$F = S(t)e^{r(T-t)} \tag{1}$$

(a) *(2 points)* Calculate the theoretical fair value of a futures contract on stock S one year before expiration, given that the current stock price is \$105 and the risk-free rate is 5% per year. Use the approximation $e^x \approx 1 + x$. \$110.25

F=$S(t)e^{r(T-t)} = 105 * e^{0.05*(t+1-t)} = 105 * e^{0.05} \approx$ 105*1.05 = \$110.25

(b) *(2 points)* The no-arbitrage principle states that in an efficient market, there should be no opportunity to make a risk-free profit without investing any capital (i.e. it should not be possible to have a portfolio with worth = \$0 at time $t < T$ and a portfolio with zero probability of being worth negative and positive probability of being positive at time T). Suppose the observed futures price is \$112. Does an arbitrage opportunity exist? If so, describe a risk-free strategy that exploits this mispricing, assuming you can borrow or invest money at the risk-free rate, buy or sell the underlying stock, and buy or sell a futures contract. Yes.
Buy one share of the underlying stock at time t for \$105, borrow \$105 from the money market at time t, and sell one futures contract expiring at time T with observed futures price of \$112 at time t.
Note the following portfolio is worth \$0 at time t, as the borrowed \$105 is used to purchase the share of the underlying stock, and the futures contract is worth \$0 at time t. Then, at time T (one year from time t), the underlying (which was bought at time t) is sold for \$112 (from selling the futures contract). Then, $\$105 * e^{0.05*1} \approx$ \$105∗1.05 = \$110.25 is owed to the money market at time T, resulting in an arbitrage profit of \$1.75.

Part C *(6 points) Option Pricing and Intrinsic Value*

(a) *(2 points)* Unlike futures, an option contract gives the right, but not the obligation, to buy (call option) or sell (put option) a stock $S(t)$ at a predetermined strike price K at a specified expiration time T. The intrinsic value of an option is the immediate profit an option holder would realize if they exercised the option at time T. Derive expressions for the intrinsic value of both call and put options at expiration. Additionally, draw a payoff diagram for both a call and put option with strike price K as a function of $S(T)$. Call Option: Max(S(T)-K,0)

Put Option: Max(K-S(T),0)

Payoff Diagram:
Call Option: For S(T) $\leq$ K, profit = 0. For S(T) > K, profit = S(T) - K (represented by a slope of 1 starting at S(T) = K, where profit is 0)
Put Option: For S(T) < K, profit = K - S(T) (represented by a slope of 1 from S(T) $\in [0, K]$). From S(T) $\geq$ K, profit = 0.

(b) *(4 points)* Consider a binomial model where stock S moves up by a factor u with probability p or down by a factor d with probability 1−p. A call option's price today is the present value of the weighted average of possible intrinsic values (based on the different possible values of $S(T)$ at expiration, where probability weights are determined by the risk-neutral probability q, satisfying:

$$r = qu + (1 - q)d \tag{2}$$

(This represents the probability of the stock going up such that the expected return of the stock is equal to the risk-free rate r). Given: $S(0) = 100$, $K = 100$, $T = 2$ periods, $u = 3\%$, d=−2%, $r = 1\%$ per period (assume for this problem, that the interest rate compounds every period, so no continuous compounding), calculate the value of the European call option today. \$2.59

As stated in the problem, a call option's price today is the PV of the weighted average of possible intrinsic values. The intrinsic value of the call option depends on the time t and the number of times the stock S moved up by a factor u (and equivalently the number of times the stock S moved down by a factor d). Denote each different possible value of the stock S as S(T,x) = $S(0) * (1+u)^x * (1+d)^{1-x}$, where T is the expiration time (or some time in the future) and x is the number of up factors in the time period from 0 to T (note 1-x is then the number of down factors in the time period from 0 to T). Then, the different possible intrinsic values of the option are represented as O(T,x) = max(S(T,x)-K,0) = max($S(0) * (1 + u)^x * (1 + d)^{1-x}$-K,0). Now, the risk free probability of the stock being in the state S(T,x) at time T is $q^x * (1 - q)^{1-x}$. Then, the PV of the weighted average of possible intrinsic values (where the interest rate compounds every period) is $(1 + r)^{-T} * \sum_{x=0}^{T} \binom{T}{x} * q^x * (1 - q)^{1-x} * O(T, x)$. Now, in this given problem, T = 2, K = 100, S(0) = 100, r = 1%, u = 3%, d = −2%. Additionally, solving the equation 1% = q*3% + (1-q)*−2% yields q = 3/5. Then, plugging these values into the above formula, the call option's price today is $(1/1.01^2)*\sum_{x=0}^{2} \binom{2}{x}*(3/5)^x*(2/5)^{1-x}*O(2,x) = (1/1.01^2)((3/5)^2*O(2,2)+2*(3/5)*(2/5)*O(2,1)+(2/5)^2*O(2,0)) = (1/1.01^2)((9/25)*max(S(2,2)-100,0)+(12/25)*max(S(2,1)-100,0)+(4/25)*max(S(2,0)-100,0) = (1/1.01^2)((9/25)*max(100*1.03^2-100,0)+(12/25)*max(100*1.03*0.98-100,0)+(4/25)*max(100*0.98^2-100,0)$ $= (1/1.0201)((9/25)*6.09+(12/25)*0.94+(4/25)*0 \approx \$2.59$.

Part D *(2 points) Early Exercise of American Options* The previous part explored European options, which only allow the holder to exercise at the expiration time T. The American option, on the other hand, allows the holder to exercise at any time $t \leq T$. Assuming no dividend payments, is it ever optimal to exercise an American call option early? If so, why, and under what conditions? (Hint: consider separately cases where $S(t) \geq K$ and $S(t) < K$ after exercising.)

No. It is never optimal to exercise an American option before expiration.

S(t) < K: If S(t) < K, then the profit from exercising early is S(t) - K < 0. However, the

intrinsic value of the call option at expiration is max(S(T)-K,0) ≥ 0, which is better than the profit from exercising early when S(t) < K for t < T.
S(t) $\geq$ K: If exercising early when S(t) $\geq$ K, then you pay K at time t and receive the stock worth S(t). On the other hand, by waiting to exercise the option at expiration, you pay K at time T>t and receive the stock worth S(T). Ultimately, in both cases, you receive the stock, but paying K dollars for the stock at time t is more costly than paying K dollars at time T due to the time value of money, and the fact that K dollars today can be invested to grow into greater than K dollars at a later time. Additionally, exercising the option early means you are giving up the time value of an option, reflected in losing out on the potential upside of the underlying in the future.

Part E *(4 points) Put-Call Parity and Arbitrage*

(a) *(2 points)* Put-call parity is a fundamental principle in options pricing that establishes a relationship between the price of a call option, a put option, the underlying asset, and a risk-free bond. It ensures that arbitrage opportunities do not exist between call and put options in efficient markets. Derive this equation by solving for the difference between a call and put option each with strike price K at time t ($c(t) - p(t)$) in terms of $S(t)$, K, and r. (Hint: Find the portfolio of the stock and bonds at time T with the same value as c(T) - p(T). What does the no arbitrage assumption say about this portfolio's value at time t?) c(t) - p(t) = S(t) - $Ke^{-r(T-t)}$

Consider c(T) - p(T) = max(S(T)-K,0) - max(K-S(T),0) = S(T) - K. Now, consider a portfolio with one share of the underlying stock and $\$K$ dollars worth of bonds at time T. Then this portfolio has value S(T) - K at time T, which is the same as the value of the portfolio with buying one call option and selling one put option at time T for any price of the underlying asset. The no arbitrage assumption says that these two portfolios must have the same value at all times t<T. Thus, at time t, the difference in the call and put options (c(t)-p(t)) is equal to S(t) - $Ke^{-r(T-t)}$.

(b) *(2 points)* Suppose at time 0, a European call and put option both with strike price \$101.50 and expiration in 6 months are priced as follows: Call price: \$6.50, Put price: \$5.50, Stock price: \$102, Interest rate: 3% per year (compounded bi-annually). Check whether put-call parity holds. If not, describe a risk-free arbitrage strategy involving the underlying asset, investing/borrowing money at the interest rate, exactly one call option, and exactly one put option, and compute the maximum arbitrage profit. Put-call parity does not hold, as c(0) - p(0) = 6.50 - 5.50 = 1 $\neq$ 2 = 102 - 101.50*1.015^{-1} = S(0) - $Ke^{-r(0.5-0)}$.

To construct a risk-free arbitrage strategy, buy one call option, sell one put option, sell one stock at \$102, and invest \$101 at the risk-free rate r. Note this portfolio is worth \$0 at time 0. Then, in six months, either exercise the call option if S(0.5) $\geq$ \$101.5, or fulfill the obligation to buy the stock from selling the put option if S(0.5) < \$101.5, closing the original short position on the stock by buying the stock for \$101.5. Then, the money invested in the money market is now worth $\$101 * 1.05 = \$102.515$, so the total arbitrage profit is $\$102.515 - \$101.5 \approx \$1.02$.

Part F *(5 points) Behavior of Option Prices*

(a) *(2 points)* How do the prices of a call and put option change as the strike price increases? Draw a graph of call and put option prices as a function of strike price K, while fixing the spot price S for some K. A call option's price decreases as K increases, as the option is less likely to be exercised, with a lower chance of S(T) being greater than K. Similarly, a put option's price increases as K increases, as the option is more likely to be exercised, with a greater chance of S(T) being lower than K. Additionally, the rate at which a call option's price declines becomes less negative as K increases. This is because an option's price is the discounted weighted average of its possible intrinsic values at expiration. If S(T) > K, then the call option will be exercised for a profit of S(T) - K, and as K increases, the profit S(T) - K becomes smaller. However, for a call option that will not be exercised (S(T) < K), then an increase in K does not change the intrinsic value of the call option (as its value remains at 0) As K increases, the amount of possibilities of S(T) where exercising the call option is optimal decreases, so the proportion of possibilities of S(T) that yield a positive intrinsic value of the call option (which decrease as K increases) also decreases. Thus, the rate at which the call option's price decreases becomes closer to zero as K increases. Similarly, the rate at which a put option's price grows is increased as K increases due to the greater number of possibilities where it is optimal to exercise the put option (S(T)<K).

This results in a non-linear, downward sloping curve for the call option, and a non-linear upward sloping curve for the put option.

(b) *(2 points)* How do the prices of a call and put option change as the underlying stock price increases? Additionally, how do the prices of a call and put option (at some time $t < T$) compare to their intrinsic value (their worth at time T)? Draw a graph of call and put option prices as a function of the underlying stock price S, while fixing strike price K or some spot price S. Compare prices at times $t < T$ and T, and explain the differences. The price of a call option increases, while the price of a put option decreases, as the underlying stock price increases. This is due to a similar explanation for part F1). When the underlying stock price increases, it increases the intrinsic value of the call option given it is optimal to exercise (and decreases for put options). Additionally, the rate of increase of a call option is greater as the underlying stock price increases, while the rate of decrease becomes less negative for put options, as an increase in the underlying stock price also increases the chance of exercising the call option being optimal (and decreases the chance for put options). This results in a non-linear, upward sloping curve for the call option, and a non-linear downward sloping curve for the put option. Additionally, the prices of both call and put options will be greater at time t<T than at time T for any value of the underlying stock price (where the time T prices are reflected in the payoff diagrams of the intrinsic value of a call and put option from part C1)). This is due to the time value of an option, which is the value that exceeds the intrinsic value of an option due to the upward potential of the underlying stock.

(c) *(1 point)* The time value of an option is the difference between its current price and intrinsic value. For a fixed strike price K, at what level of the underlying stock price S does the time value of an option reach its maximum? Explain intuitively why this occurs. The time value of an option is highest when the underlying stock price is near the strike price K. This is because there is the most uncertainty about whether exercising the option will lead to a positive payoff at expiration (when S(T)>K for call options or S(T)<K for put options), which increases the potential value of the option over time.

Problem 2: The Tariff-ic Game (25 points)

In this problem, we explore a game theoretic approach to the ramifications of tariffs.

Preliminaries Consider two identical countries. Each country has a government, a firm and consumers who buys from the two firms. (Assume the governments, firms and consumers are completely identical in each country, and that the firms produce identical goods.) In this game, governments first choose to levy a tariff, denoted t_i. Then, firms choose how much they want to produce for domestic consumption and exports, denoted h_i and e_i, respectively. Country i's firm has a market demand curve given by $P_i = a - h_i - e_j$, where h_i denotes the quantity produced for home consumption and e_j denotes the quantity exported to country i from country j.

Participants were expected to show work throughout this question. For certain parts, points may have been deducted regardless of whether a team reached the correct answer if insufficient work was shown.

Part A *(2 points)* Suppose both firms have a constant marginal cost, c. What is the total cost function for firm i? Firm j?

For firm i, the total cost function is $c(h_i + e_i) + t_j e_i$. This is the sum of the variable cost and the cost induced by the tariff on firm i's exports. The cost function for firm j is identical, but with swapped subscripts. One point was awarded for correct expression of total quantity produced by the firm, another point was awarded for the correct answer.

Part B *(1 point)* What is the profit function (denoted π_i) for each firm?

The total revenue for firm i is $P_i(h_i) + P_j(e_i)$, which becomes $(a - h_i - e_j)(h_i) + (a - h_j - e_i)(e_i)$. Combined with the total cost function, we have $\pi_i = (a - h_i - e_j)(h_i) + (a - h_j - e_i)(e_i) - c(h_i + e_i) - t_j e_i$. One point was awarded for having the correct profit function.

Part C *(2 points)* Each government seeks to maximize its total welfare function, which can be expressed as the sum of consumer surplus, producer surplus, and revenue collected from the tariff. What is the welfare function for government i?

The consumer surplus is $\frac{1}{2}(h_i + e_j)^2$, the profit function is the same as in Part B, and the revenue from the tariff is $t_i e_j$. (Note: to easily see where the expression for consumer surplus comes from, consider graphing a simple graph with the demand curve and a horizontal line for price. The area bounded by these lines and the y-axis is the consumer surplus.) Putting these together, we get $W_i = \frac{1}{2}(h_i + e_j)^2 + (a - h_i - e_j)(h_i) + (a - h_j - e_i)(e_i) - c(h_i + e_i) - t_j e_i + t_i e_j$. One point was awarded for the correct consumer surplus and tariff. One point was awarded for a correct answer.

Part D *(3 points total)* Suppose government i levies a fixed tariff of t_i^* and government j levies a fixed tariff of t_j^*. Further suppose firm j fixes quantities h_j^* and e_j^*. Compute the profit maximizing quantities, h_i^* and e_i^*, chosen by firm i as a function of t_i^*, t_j^*, h_j^*, e_j^* and constants. *(Hint: consider breaking this problem into two separate maximization problems.)*

We are maximizing the function $(a - h_i - e_j^*)(h_i) + (a - h_j^* - e_i)(e_i) - c(h_i + e_i) - t_j^* e_i$ with respect to h_i and e_i. We can separate this expression into the sum of two expressions: one in terms of h_i and one in terms of e_i. So, it suffices to maximize each expression with respect to its associated variable. So, we maximize $(a - h_i - e_j^* - c)(h_i)$ with respect to h_i and $(a - h_j^* - e_i - c)(e_i) - t_j^* e_i$ with respect to e_i. This yields first-order conditions of $a - 2h_i - e_j^* - c = 0$ and $a - h_j^* - 2e_i - c - t_j^* = 0$. Solving yields $h_i^* = \frac{1}{2}(a - e_j^* - c)$ and $e_i^* = \frac{1}{2}(a - h_j^* - c - t_j^*)$. One point was awarded for having the correct initial setup. One point was awarded for correct first-order conditions. One point was awarded for having correct answers.

Part E *(1 point)* Now compute the profit maximizing quantities for firm j as a function of t_i^*, t_j^*, h_i^*, e_i^* and constants. *(Hint: the firms are symmetric!)*

Since the firms are symmetric, we can take a shortcut and just swap subscripts: $h_j^* = \frac{1}{2}(a - e_i^* - c)$ and $e_j^* = \frac{1}{2}(a - h_i^* - c - t_i^*)$. Otherwise, the math for this part and Part D is exactly the same, just with swapped subscripts. One point was awarded for having a correct answer.

Part F *(2 points)* Solve the resulting system of equations for h_i^*, e_i^*, h_j^*, and e_j^* as a function of the fixed tariffs and constants.

Solving the system of equations using the expressions in Parts D and E yields for firm i, $h_i^* = \frac{a-c+t_i^*}{3}$, $e_i^* = \frac{a-c-2t_j}{3}$. By symmetry, the exact same quantities are chosen by firm j, but with swapped subscripts. One point was awarded for having a correct expression for h. One point was awarded for having a correct expression for e.

Part G *(1 point)* Rewrite the welfare function for government i in terms of tariffs, t_i and t_j, and constants.

We substitute the values from Part F into the expression determined in Part C. This yields $W_i = \frac{(2(a-c)-t_i)^2}{18} + \frac{(a-c+t_i)^2}{9} + \frac{(a-c-2t_j)^2}{9} + \frac{t_i(a-c-2t_i)}{3}$. One point was awarded for having the correct answer. No points were deducted for having an equivalent correct answer.

Part H *(2 points)* Assume government j fixes a tariff of t_j^*. Solve for the welfare-maximizing tariff t_i^* chosen by government i.

We maximize the welfare function in Part G with respect to t_i, taking t_j as a constant. This yields a first-order condition of $-\frac{2(a-c)-t_i}{9} + \frac{2(a-c+t_i)}{9} + \frac{a-c-4t_i}{3} = 0$. Solving yields $t_i^* = \frac{a-c}{3}$, One point was awarded for having the correct first-order condition. One point was awarded for having the correct answer.

Part I *(1 point)* What is the welfare-maximizing tariff t_j^* chosen by government j assuming government i fixes a tariff of t_i^*? *(Hint: the governments are also symmetric!)*

Just like in Part E, we note that the governments of country i and country j are symmetric, so the tariff chosen by government j is the same as the tariff chosen by government i, that is $t_j^* = \frac{a-c}{3}$. One point was awarded for having a correct answer.

Part J *(1 point)* Using the tariffs you found in Part I, solve for the quantities produced by firms i and j in terms of constants.

We substitute the tariff values we got in Parts H and I into the expressions we determined from Part F. This yields $h_i^* = h_j^* = \frac{4(a-c)}{9}$, $e_i^* = e_j^* = \frac{a-c}{9}$. One point was awarded for having a correct answer.

Part K *(2 points)* Show that having zero tariffs maximizes total welfare across both countries.

We wish to maximize the sum of the welfare functions for the two countries with respect to the tariffs each country imposes. That is, we are maximizing $W_i + W_j = \frac{(2(a-c)-t_i)^2}{18} + \frac{(a-c+t_i)^2}{9} + \frac{(a-c-2t_j)^2}{9} + \frac{t_i(a-c-2t_i)}{3} + \frac{(2(a-c)-t_j)^2}{18} + \frac{(a-c+t_j)^2}{9} + \frac{(a-c-2t_i)^2}{9} + \frac{t_j(a-c-2t_j)}{3} =$ with respect to t_i and t_j. Just like in Part D, we can separate this problem into two separate maximization problems. These are: maximizing $\frac{(2(a-c)-t_i)^2}{18} + \frac{(a-c+t_i)^2}{9} + \frac{(a-c-2t_i)^2}{9} + \frac{t_i(a-c-2t_i)}{3}$ with respect to t_i, and maximizing $\frac{(2(a-c)-t_j)^2}{18} + \frac{(a-c+t_j)^2}{9} + \frac{(a-c-2t_j)^2}{9} + \frac{t_j(a-c-2t_j)}{3}$ with respect to t_j. This yields first-order conditions of $-\frac{2(a-c)-t_i}{9} + \frac{2(a-c+t_i)}{9} - \frac{4(a-c-2t_i)}{9} +$

$\frac{a-c-4t_i}{3} = 0$ and $-\frac{2(a-c)-t_j}{9} + \frac{2(a-c+t_j)}{9} - \frac{4(a-c-2t_j)}{9} + \frac{a-c-4t_j}{3} = 0$. This yields $\frac{t_i}{9} = -\frac{a-c}{9}$, which simplifies to $t_i = -(a-c)$. This implies the optimal choice for tariffs is $t_i = t_j = -(a-c)$. However, this implies the governments should set negative tariffs, so the welfare-maximizing level of tariffs is 0 (the first-order conditions are decreasing in t_i and t_j. One point was awarded for correct setup, one point was awarded for sufficient proof (i.e. showing the answer to the maximization problem is indeed having 0 tariffs).

Part L *(7 points total)* In the famous Cournot competition model, two firms in the same market simultaneously choose a quantity q to produce. The demand curve for both firms is given by $P = a - q_i - q_j$, where q_i and q_j denote the quantities chosen by each firm.

(a) *(2 points)* Solve for the profit maximizing quantity chosen by each firm as a function of the quantity produced by the other firm.
The profit function for each firm is $\pi_i = (a - q_i - q_j^* - c)q_i$. This yields first-order conditions of $a - 2q_i - q_j^* - c = 0$, which simplifies to $q_i = \frac{a-q_j^*-c}{2}$. The expressions for the other firm have swapped subscripts (because of symmetry). One point was awarded for each correct quantity expression.

(b) *(1 point)* Solve the system of equations for the profit maximizing quantity chosen by each firm. Solving the system of equations yields $q_i^* = q_j^* = \frac{a-c}{3}$. One point was awarded for reaching the correct answer.

(c) *(4 points)* Compare the outcome of the Cournot game to the outcome of the tariffs game. How does the quantity produced by each firm differ? What about market price? In which game are consumers better off? Producers? Quantity in the tariff game is $h_i + e_j = \frac{5(a-c)}{9}$ while quantity in the Cournot game is $q_i + q_j = \frac{2(a-c)}{3}$. Therefore, while the total quantity in the world is higher in the tariff game, the quantity available to consumers in each market is higher in the Cournot game. Since price is given by $a - Q$, where Q is the total quantity on the market, the prices for consumers are higher in the tariff game than in the Cournot game (where the price is exactly marginal cost, c). For welfare analysis, we compare the consumer and producer surplus (profit) in each game. Consumer surplus in both games was $\frac{1}{2}Q^2$. Since Q is higher in the Cournot game, consumer surplus is thus higher in the Cournot game. So, we conclude consumers are better off in the Cournot game. Profit in the Cournot game is $\frac{(a-c)^2}{9}$. Profit in the tariffs game is $\frac{17(a-c)^2}{81}$. Clearly, profits in the tariffs game are much higher. So, producers are better off in the tariffs game. One point each was awarded for correctly identifying the higher value for quantity, price, consumer surplus and producer surplus with sufficient mathematical justification.

Problem 3: Weird Guy at Birthday Party (25 points)

In this problem, we consider ways to divide goods among multiple individuals so that everyone is satisfied. Various notions of what this means in particular will be elaborated on and explored.

Preliminaries We begin by defining two potential criteria for fair division: *proportionality* and *envy-freeness*. Suppose for $i \in \{1, 2, ..., n\}$, each agent i receives a share worth X_i, where the entire cake is worth $1 = \sum_{i=1}^{n} X_i$.

(a) We say the division is *proportional* if for all $i \in \{1, 2, ..., n\}$, $X_i \geq 1/n$.
I.e. each agent receives at least $1/n$ of the whole cake.

(b) We say the division is *envy-free* if for all i, $j \in \{1, 2, ..., n\}$, $X_i \succeq_i X_j$.
I.e. each agent weakly prefers their share to any other share.

Part A *(2 points total)* Suppose there are 2 agents, Azir and Bard, attempting to split a cake. Define the cake as the interval $[0, 1]$, where each agent values their piece based on the length of the line segment they receive. We can make a proportional and envy-free division by using the following process. First, Azir chooses a cut $k \in [0, 1]$ where he prefers both sides equally. Then, Bard chooses either the left $[0, k]$ or right piece $(k, 1]$.

(a) *(1 point)* Explain why this division is proportional.

(b) *(1 point)* Explain why this division is envy-free.

(a) Azir gets at least 1/2, because he chooses to cut into 2 equal sides. Bard chooses the larger piece (at least one of which must be at least 1/2), so he gets at least 1/2.

(b) Azir guarantees that both sides are 1/2 in his view. Bard guarantees that his piece is at least 1/2 because he chooses the larger piece.

One point was given for justifying why the division is proportional, and one point was given for justifying why the division is envy-free.

Part B *(1 point)* An additional criteria we may consider is *symmetry*. We say the division is *symmetric* if every agent is guaranteed the same value no matter the permutation of roles the agents receive. Explain why the division from Part A is not symmetric.

As the "chooser," Bard can guarantee he gets at least 1/2. However, Azir will never get more than 1/2 as the "cutter." *One point was given for justifying why the division is not symmetric.*

Part C *(2 points)* Consider the following process. Azir and Bard each make a mark where they believe the halfway point is at a and b respectively. Then a cut is made at $(a + b)/2$. Explain how the process should continue to ensure a symmetric, proportional, envy-free division.

Azir gets the side a is on and Bard gets the other piece. *Two points were given for an appropriate process.*

Part D *(3 points)* Suppose we have the cake from $[0, 1]$ as defined in Part A. However, there is a 3rd person, Caitlyn, who wishes to eat the cake too. We want to define a proportional and envy-free division. We proceed in the following way:

(a) Azir divides the cake into thirds, creating pieces X, Y, Z.

(b) Suppose Bard believes X is the biggest piece. Then, Bard trims X to be the same as the 2nd largest piece. Let X^L be the larger trimmed piece of X and let X^s be smaller trimming of X.

(c) Set aside X^s to be divided later. We proceed with dividing X^L, Y, and Z.

(d) Caitlyn chooses between X^L, Y, and Z.

(e) If Caitlyn did not take X^L, Bard must take X^L. Otherwise, Bard chooses from the remaining two pieces.

(f) Azir takes the last piece.

Suppose after Step 1, Bard believes the two largest pieces are the same size. Describe how we can continue the process to ensure the division is proportional and envy-free. Justify why the division meets those criteria.

The agents choose each of the 3 pieces in the order of: Caitlyn, Bard, Azir. Caitlyn gets first pick, so she gets the (weakly) largest piece. Bard thinks the two largest pieces are the same size, so at least one of these largest pieces will be available for him to pick. Azir made the cuts, so he can ensure all 3 pieces is 1/3 (and therefore his piece is at least as good as the others). *Two points were given for the correct process. One point was given for justifying that the process is proportional and envy-free.*

Part E *(6 points total)* Suppose after Step 1, Bard believes there is one piece that is strictly larger than the others. Then, we proceed with Step 2-6. We have divided up X^L, Y, and Z, but the trimming X^s still remains to be divided.

(a) *(3 points)* Describe how we can continue the process to ensure the division is proportional and envy-free. (Hint: let Bard or Caitlyn, whoever did not choose X^L, make cut(s). Then, come up with an ordering of "choosers".)

(b) *(3 points)* Justify that your division in Part E is envy-free.

(a) Suppose without loss of generality that Bard has X^L. Then, let Caitlyn cut X^s into equal thirds. Finally, let the agents choose in the order of: Bard, Azir, Caitlyn. We will denote these pieces as X^s_B, X^s_A, X^s_C respectively.

(b) Bard has X^L and X^s_B. Bard guarantees $X^L \geq Y$ and $X^L \geq Z$, because Bard cut X^L such that X^L is one of the 2 largest out of X^L, Y, Z. (If instead Caitlyn had X^L, it is still the case that $X^L \geq Y$ and $X^L \geq Z$, because Caitlyn chose between X^L, Y, Z in step 4). Bard chose first out of X^s_B, X^s_A, X^s_C, so $X^s_B \geq X^s_A$ and $X^s_B \geq X^s_C$.
Without loss of generality (it does not matter if Caitlyn has Y or Z), Caitlyn has Y and X^s_C. $Y \geq X^L$ and $Y \geq Z$, because Caitlyn chose between X^L, Y, Z in step 4. (If instead Bard had Y, it is still the case that $Y \geq X^L$ and $Y \geq Z$. Note that at least one of Y or Z is at least as big as X^L, since Bard cut X^L such that this is true. Bard chose Y over Z in step 5, so $Y \geq Z$ and therefore $Y \geq X^L$.) Caitlyn cut X^s_B, X^s_A, X^s_C, so all parts are equal to her.
Azir has Z and X^s_A. First, note that $Z + X^s_A \geq X^L + X^s_B$, because Azir's initial cuts into thirds guarantees $Z + X^s_A \geq Z = X \geq X^L + X^s_B$. Then, Azir's initial cuts also guarantees $Z = Y$. Finally, $X^s_A \geq X^s_C$ since Azir chose between X^s_A and X^s_C. (Notice that this relies on the fact that Azir is guaranteed to have Y or Z and not X^L. This is guaranteed by the fact that in step 5, Bard must choose X^L if Caitlyn did not already take it.)

Three points were given for an appropriate process. Three points were given for proper justification of envy-freeness. Two points were given for (2) if there were minor omissions in the justifications.

Part F *(3 points total)* Suppose Azir, Bard, and Caitlyn receive another cake they want to divide. This time, the cake is 2-dimensional. First, each of the three makes a vertical mark so that the piece to the left of the mark is worth $1/3$. Suppose without loss of generality that Azir made the leftmost mark. Then, Azir receives the left piece.

(a) *(1 point)* Suppose we divide the right piece by using the division from Part A. Explain why Bard and Caitlyn do not envy anyone.

(b) *(2 points)* Explain why the process as a whole is not envy-free.

(a) Suppose without loss of generality that we assign Bard to be the "cutter" and Caitlyn to be the "chooser." By the same justification in Part A (2), they do not envy each other. Next, notice Bard and Caitlyn believe Azir's piece is at most $1/3$ (since their marks were to the right of Azir's). Thus, Bard gets to split the remaining piece (which is at least $2/3$) into halves, so he does not envy Azir. Caitlyn gets to choose between the pieces Bard cuts, so that piece is at least $1/3$, so Caitlyn does not envy Azir.

(b) The problem is that Azir cannot guarantee that Bard and Caitlyn do not split the remaining piece (which he views as $2/3$) exactly in half such that Bard and Caitlyn each get $1/3$. Suppose, for example, that Bard cuts the remaining $2/3$ into $1/6$ and $1/2$. Then, Caitlyn could choose the $1/2$ piece and have a larger piece than Azir's $1/3$. Thus, the process is not envy-free.

One point was given for explaining why Bard and Caitlyn do not envy anyone. Two points were given for explaining why Azir may envy Bard or Caitlyn.

Part G *(8 points total)* Instead of splitting the remaining right piece using the division from Part A, we will proceed as follows. Azir will rotate his knife over the cake for each angle $\theta \in [0, \pi]$ such that he believes each side is exactly $1/2$. (It may help to imagine the remaining piece as a circle, although the specific shape is unimportant to the problem).

(a) *(1 point)* Let B be Bard's valuation of the remaining piece. Consider $X_b(\theta)$, Bard's valuation of the piece left of Azir's knife at angle θ. Express $X_b(\pi)$ in terms of $X_b(0)$, B.

(b) *(3 points)* Explain why there is guaranteed to be an angle θ for which $X_b(\theta) = \frac{1}{2}B$.

(c) *(2 points)* Describe how the process should continue to guarantee it is envy-free.

(d) *(2 points)* Justify that the process is envy-free.

(a) Notice that the piece to the right of the knife at θ is $B - X_b(\theta)$. Then, notice that the piece to the left of the knife at angle π (valued at $X_B(\pi)$) is the piece to the right of the knife at angle 0. Thus, $X_b(\pi) = B - X_b(0)$.

(b) Case 1: $X_b(\pi) = B - X_b(0) \leq X_b(0)$. Adding $X_b(0)$ to both sides and dividing by 2, $0.5B \leq X_b(0)$. Then, $X_b(\pi) = B - X_b(0) \leq 0.5B \leq X_b(0)$.
Case 2: $X_b(\pi) = B - X_b(0) \geq X_b(0)$. Adding $X_b(0)$ to both sides and dividing by 2, $0.5B \geq X_b(0)$. Then, $X_b(\pi) = B - X_b(0) \geq 0.5B \geq X_b(0)$.
Notice that Azir rotates his knife over the cake continuously on the interval $\theta \in$

$[0, \pi]$. Since Bard's valuation is simply the area of the cake, his valuation $X_b(\theta)$ is a continuous function on the domain $[0, \pi]$. Since $X_b(\pi) \leq 0.5B \leq X_b(0)$ or $X_b(\pi) \geq 0.5B \geq X_b(0)$, by the Intermediate Value Theorem, there exists some θ^* such that $X_b(\theta^*) = \frac{1}{2}B$.

(c) Bard tells Azir when to cut such that $X_b(\theta) = \frac{1}{2}B$. Then, Caitlyn chooses between those 2 pieces.

(d) In Part F (1), we concluded that Bard and Caitlyn believe Azir's piece is at most 1/3. Since Caitlyn chooses between the 2 remaining pieces (which are altogether at least 2/3), Caitlyn can guarantee that her piece is at least 1/3. Bard chooses where to cut the remaining piece (which is at least 2/3) so that each piece is equal, so Bard can guarantee his piece is at least ($\frac{2}{3}/2 = 1/3$). Thus, Caitlyn and Bard do not envy Azir. Caitlyn and Bard do not envy each other, because Caitlyn chooses the larger of the 2 remaining pieces and Bard cut the 2 remaining pieces to be equal. Finally, Azir does not envy Caitlyn or Bard, because his piece is at least 1/3. The remaining piece is therefore at most 2/3. Since Azir rotated the knife such that both sides are exactly 1/2 of the remaining piece, both sides are at most $\frac{2}{3}/2 = 1/3$.

One point was given for the correct expression for (1). Two points were given for justifying that 0.5B is between $X_b(\pi)$ and $X_b(0)$ for (2). One point was given for correctly justifying that there exists θ such that $X_b(\theta) = \frac{1}{2}B$ using the IVT for (2). Two points were given for providing the correct process for (3). Two points were given for justifying envy-freeness for (4).

Problem 4: When YAKs Get Smarter (25 Points)

In this problem we consider how the growth rate of technologies usefulness can impact the returns to labor and capital by building on a basic growth model. This is a model where learning by doing is possible.

Preliminaries This question concerns a country, called Yakland, with Cobb Douglas growth and interchangeable human capital and physical capital. In this question, consider upper case letters to represent total values while lower case letters represent per capita values. Thus Y represents output while y represents output per capita or $y = \frac{Y}{L}$, where L is the population, and K represents capital while k is capital per capita.

Part A *(1 Point)* In this economy output is the product of capital, K, and the productivity of capital, denoted A. Write this relationship as an equation. This equation is simply $Y = AK$. One point for the correct answer

Part B *(1 Point)* Solve this equation for the per capita values of output. Do not include L in the equation. The relationship holds even in per capita terms so the correct answer is $y = Ak$. One point for correct answer

Part C *(2 Points)* Assume that capital depreciates at the rate δ and there is no growth in population. Consumer save a portion of their income, denoted s. These savings are invested into physical capital. Find an equation for the next period's capital, denoted K_{t+1} as a function of present capital K_t, depreciation, the savings rate and the current period's output denoted Y_t. The equation is described and can be represented as $K_{t+1} = sY_t + (1 - \delta)K_t$. 2 points for correct answer

Part D *(1 Point)* Find the growth equation of capital per capita. Remember there is no growth in population. Again, these values do not change when represented per capita because there is no multiplication of per capita terms. $k_{t+1} = sy_y - \delta k_t$

Part E *(1 Point)* A country is at a steady state if there is no change capital from one period to another. Find the condition required for Yakland to be at a steady state. Remember that output is function of capital as well. We can solve this by seeing when $k_{t+1} = k_t$. Filling in the appropriate terms for y_t lets us find that $1 = sA - \delta$. This is when the steady state is reached. One point for correct answer

Part F *(2 Points)* Suppose we have a country with N firms. We denote the production of each firm

$$y_i = \bar{A}k_j^{\alpha}L_j^{1-\alpha}$$

where

$$\bar{A} = A_0(\sum_{i=1}^{N} k_j)^{\tau}$$

Describe how a firm's decision to increase capital effects $\bar{A}$ and what that means for the utilization of capital. The important thing to note here is that increasing investment also increases the marginal productivity of investments. This creates a positive externality as one firm investing increases the returns of all other firms. One point was awarded for noticing the direction of the relationship and one point for describing the externality.

Part G *(1 Points)* Assume the firms are symmetric and normalize L_j to 1. Represent the countries total capital, K as a function of k_j and N. Represent the country's total output, Y as a function of y_j and N. Since the firms are symmetric we can simply say the market is the sum of firms and total output is the sum of total outputs. Thus, $K = Nk_j$ and $Y = Ny_j$. One point for correct answer.

Part H *(2 Points)* Use the prior question to find $\bar{A}$ in terms of non-firm (without indices) variables. We can replace the parantheses in with K since it equals Nk_j. Thus we get $\bar{A} = A_0 K^{\tau}$. Two points for correct answer.

Part I *(1 Point)* Now represent the country's total output only as a function of non-firm variables. We can plug in what we have for $\bar{A}$ and notice multiplying each side by N gives the function $Y = A_0 K^{\tau+\alpha}$. At this point you should recognize the similarities with the $Y = AK$ model discussed before. One point awarded for correct answer.

Part J *(2 Points)* Recall that $K_{t+1} = sY - \delta K$. Use that the law of capital growth is the same as found in part C (not in part D). Under what conditions does this function operate equivalently to the model discussed in Part E? Show the most simplified equivalence. Plugging our new model in to the equation allows us to find that the impact of the new model is based on $\bar{A}$ and the sum of $\tau + \alpha$. Breaking those down lets us see the important chains are that N does not change and that $\tau + \alpha$ is equal to one. Two points were awarded for this second explanation but given the similarities with the next part, full credit could be earned if this relationship was later mentioned.

Part K *(3 Points)* Under what condition will the steady state converge to the same equation as part E? Under what condition will this economy experience explosive growth? Demonstrate how the economy will act when above or below the steady state. The critical explanations are that the economy can reach a steady state when $\tau + \alpha \leq 1$ but that the economy will experience explosive growth if the coefficient on $\tau+\alpha > 1$. If in a steady state condition, the market will return to the steady state asymptotically if it has a deviation. 1 point for each inequality and 1 point was awarded for understanding stead state dynamics.

Part L *(2 Points)* Find an equation for the steady state of capital, denoted K^* under the steady state condition. Similarly to in early parts, we must find a K for which $K_{t+1} = K_t$. Plugging in the new definitions for Y gives the relationship that $K = (\frac{sA_0}{1+\delta})^{\frac{1}{1-\tau-\alpha}}$. This was one interpretation based on the question. Another interpretation is that $\delta K = sY$ in which case we get the answer that $(\frac{\delta}{sA_0})$

$1\alpha + \tau - 1$=K. Both are accepted for 2 points due to the phrasing of the question. The more correct interpretation would be the second.

Part M *(2 Points each)* On 3 separate graphs consider an economy in the steady state. Demonstrate how the level of capital changes before and after each shock (independently).

- *(2 Points)* An unexpected and permanent increase in the savings rate Graph these with K^* on the Y-axis and t on the X-axis. This graph will show a flat K^* that then increases permanently approaching the steady state asymptotically. This graph would receive full credit but partial credit is available if parts of the graph are correct so long as it was interpretable. It is also possible to make an argument that K^* falls using golden rule consumption but that's a more difficult and unintuitive argument.
- *(2 Points)* An unexpected inflow of foreign aid in the form of capital This will cause a temporary increase in the capital shown as a one time shock that then asymptotically

decreases back to the steady state level.

– *(2 Points)* The unexpected exit of a significant number of firms from the market to other countries Losing firms is the same as a permenant loss of capital in this market which would indicate a decrease in the output and in the steady state level of capital. Show this the same way on the graph. Each part is graded and scored the same way.

Problem 5: Pricing Perfect Purple Pens (24 Points)

In this problem, we analyze how firms set prices when dealing with homogeneous products in limited and infinite periods.

Preliminaries Suppose there is a market in Greendale where two firms (Firm A and Firm B) produce purple pens. In this market, assume that each firm simultaneously sets a price for purple pens. Consumers then decide how many purple pens to buy and which firm to buy from.

Part A Suppose that Firm A and Firm B produce homogenous (i.e. identical) products. In this market, if one price is less than the other price, consumers only buy purple pens from the lower priced firm. If both prices are the same, each firm sells half of the total output demanded at that price. Total demand at the purchased price is determined by the demand curve listed below. Assume α and β are positive constants and each firm has a unit cost of production of c:

$$P = \alpha - \beta * Q$$

(2 Points) In this market, write the respective profit functions for Firm A and Firm B as functions of α and β. (Hint: the profit functions are discontinuous.)

Firm A Profit Function:

$$P_A < P_B : \pi_A = \frac{(P_A - c)(\alpha - P_A)}{\beta}$$
$$P_A = P_B : \pi_A = \frac{(P_A - c)(\alpha - P_A)}{2\beta}$$
$$P_A > P_B : \pi_A = 0$$

Firm B Profit Function:

$$P_A > P_B : \pi_B = \frac{(P_B - c)(\alpha - P_B)}{\beta}$$
$$P_A = P_B : \pi_B = \frac{(P_A - c)(\alpha - P_A)}{2\beta}$$
$$P_A < P_B : \pi_B = 0$$

If $P_A < P_B$, Firm A takes the entire market share and earns a profit selling at price P_A and quantity Q. If $P_A = P_B$, Firm A and Firm B split the market and split the profit selling at a price P_A and quantity Q evenly (thus the profit is half of when $P_A < P_B$). If $P_A > P_B$, Firm B takes the entire market share and Firm A earns a profit of 0. Firm B has symmetrical payoffs and its profit function can be derived using the same logic.

1 point awarded for partial accuracy on the profit functions and 2 points awarded for full accuracy.

Part B *(2 Points)* Given the above profit functions, what is the unique Nash equilibrium price for Firm A and Firm B. (Hint: the Nash equilibrium price for Firm A and Firm B will be the price such that neither firm has the incentive to switch prices after knowing the price the other firm has set.)

The unique Nash equilibrium is $P_A = P_B = c$. Both firms will keep slightly undercutting each others' prices until price reaches the marginal cost of production c, during which there is no incentive to deviate as increasing prices leads to 0 profits and decreasing prices leads to negative profit.

1 point awarded for $P_A = P_B$ at the Nash equilibrium. 1 point awarded for finding $P_A = P_B = c$.

Part C *(1 Point)* Given the Nash equilibrium prices, what are the profits for Firm A and Firm B?

$\pi_A = \pi_B = 0$. Price equals marginal cost c for both firms so profit must equal 0 for both firms as well.

0.5 points for correct setup in calculating profits. 0.5 points for finding $\pi_A = \pi_B = 0$.

Part D Now suppose that Firm A and Firm B **don't** produce homogenous products (i.e. their products are differentiated). Suppose that the two firms sell products whose demand curves are listed below. Assume that α and β are positive constants and that each firm has **zero costs of production**.

$$\text{Firm A: } q_1 = \alpha - \beta * p_1 + x(p_2 - p_1)$$
$$\text{Firm B: } q_2 = \alpha - \beta * p_2 + x(p_1 - p_2)$$

(2 Points) In this market, write the respective profit functions for Firm A and Firm B as functions of α, β, p_1, p_2, and x.

$$\pi_A = \alpha p_1 - \beta p_1^2 + x p_1 p_2 - x p_1^2$$

$$\pi_B = \alpha p_1 - \beta p_1^2 + x p_1 p_2 - x p_1^2$$

Find the respective profits simply by calculating $\pi_A = p_1 * q_1$ and $\pi_B = p_2 * q_2$. 2 points awarded for correct profits.

Part E *(2 Points)* Given the above profit functions, what is the unique Nash equilibrium price for Firm A (p_1) and Firm B (p_2) in terms of x. (Hint: set the first derivatives of the profit functions equal to zero).

Take the derivative of the profit functions found from part D and set the equation equal to 0, solving for p_1 and p_2 to find the price that maximizes profit for Firm A and Firm B, respectively.

$$\frac{d\pi_A}{dp_1} = \alpha - 2\beta p_1 + x p_2 - 2x p_1 = 0$$

$$2\beta p_1 + 2xp_1 = \alpha + xp_2$$
$$p_1 = \frac{\alpha + xp_2}{2\beta + 2x}$$

$$\frac{d\pi_B}{dp_2} = \alpha - 2\beta p_2 + xp_1 - 2xp_2 = 0$$
$$2\beta p_2 + 2xp_2 = \alpha + xp_1$$
$$p_2 = \frac{\alpha + xp_1}{2\beta + 2x}$$

$$p_1 = \frac{\alpha}{2\beta + 2x} + \frac{x}{2\beta + 2x} \cdot \frac{\alpha + xp_1}{2\beta + 2x}$$
$$p_1 = \frac{\alpha}{2\beta + 2x} + \frac{\alpha x + x^2 p_1}{(2\beta + 2x)^2}$$
$$(2\beta + 2x)^2 p_1 = 2\alpha\beta + 2\alpha x + \alpha x + x^2 p_1$$
$$p_1 = \left[(2\beta + 2x)^2 - x^2\right] p_1 = 2\alpha\beta + 3\alpha x$$
$$p_1 = \frac{2\alpha\beta + 3\alpha x}{(2\beta + 2x)^2 - x^2}$$
$$p_1 = \frac{\alpha(2\beta + 2x + x)}{(2\beta + 2x + x)(2\beta + 2x - x)}$$
$$p_1 = \frac{\alpha}{2\beta + x}$$

$$p_2 = \frac{\alpha}{2\beta + 2x} + \frac{x}{2\beta + 2x} \cdot \frac{\alpha + xp_2}{2\beta + 2x}$$
$$p_2 = \frac{\alpha}{2\beta + 2x} + \frac{\alpha x + x^2 p_2}{(2\beta + 2x)^2}$$
$$(2\beta + 2x)^2 p_2 = 2\alpha\beta + 2\alpha x + \alpha x + x^2 p_2$$
$$\left[(2\beta + 2x)^2 - x^2\right] p_2 = 2\alpha\beta + 3\alpha x$$
$$p_2 = \frac{2\alpha\beta + 3\alpha x}{(2\beta + 2x)^2 - x^2}$$
$$p_2 = \frac{\alpha(2\beta + 2x + x)}{(2\beta + 2x + x)(2\beta + 2x - x)}$$
$$p_2 = \frac{\alpha}{2\beta + x}$$

$$\text{Nash Equilibrium} = \left(\frac{\alpha}{2\beta + x}, \frac{\alpha}{2\beta + x}\right)$$

1 point for finding $p_1 = p_2$. 1 point for successfully solving the derivatives and the values of p_1 and p_2.

Part F *(1 Point)* Given the Nash equilibrium prices, what are the profits for Firm A and Firm B?

$$\pi_A = \alpha p_1 - \beta p_1^2 + x p_1 p_2 - x p_1^2$$

$$\pi_A = \frac{\alpha^2}{2\beta + x} - \beta \cdot \frac{\alpha^2}{(2\beta + x)^2} + x \cdot \frac{\alpha^2}{(2\beta + x)^2} - x \cdot \frac{\alpha^2}{(2\beta + x)^2}$$

$$\pi_A = \frac{\alpha^2(2\beta + x) - \beta\alpha^2}{(2\beta + x)^2}$$

$$\pi_A = \frac{\alpha^2(\beta + x)}{(2\beta + x)^2}$$

$$\pi_B = \alpha p_2 - \beta p_2^2 + x p_1 p_2 - x p_2^2$$

$$\pi_B = \frac{\alpha^2}{2\beta + x} - \beta \cdot \frac{\alpha^2}{(2\beta + x)^2} + x \cdot \frac{\alpha^2}{(2\beta + x)^2} - x \cdot \frac{\alpha^2}{(2\beta + x)^2}$$

$$\pi_B = \frac{\alpha^2(2\beta + x) - \beta\alpha^2}{(2\beta + x)^2}$$

$$\pi_B = \frac{\alpha^2(\beta + x)}{(2\beta + x)^2}$$

Solve for profits by plugging in the values of p_1 and p_2 found in Part E into their respective profit functions. 0.5 points for correct logic/process. 0.5 points for correct final profits.

Part G *(1 Point)* Do increases in x cause prices to fall or rise? Provide an economic explanation for this effect.

Increases in x will cause prices to fall. x represents cross-price elasticity, so as x increases, the goods become more substitutable (or less differentiated), causing prices to decrease. 0.5 points for correctly stating that an increase in x causes prices to fall. 0.5 points for the correct economic explanation, involving either cross-price elasticity or differentiation of goods.

Part H *(2 Points)* Compare the Nash equilibrium profits and prices for both firms in Part A and B and provide an economic explanation for the difference.

The difference is that there are no profits in the first part with homogeneous products. Because the products both firms are selling are identical, prices will inevitably be driven down to c, leading to zero profits. In the second part, however, there is product differentiation, leading to the possibility of positive profits. 1 point for correctly identifying 0 profits vs. positive profits. 1 point for correctly stating the economic explanation, involving product differentiation.

Part I/J *(7 points)*Consider the market given in Part A and assume the same conditions (both firms simultaneously announce a price and if one of the prices is lower than the other, the firm charging the lower price sells the amount demanded at that price, and the firm charging the higher price sells nothing). Now assume that the firms play the following stage game an infinite number of times, where both firms simultaneously announce a price at the beginning of each period. Assume that both firms have a discount rate of $0 < \delta < 1$ in each period. Considering price can be any value between c and α, what is the minimum discount rate δ in order for there to exist a Nash equilibrium where both firms charge a price of $p^* = (\alpha - c)/2\beta$ in each period?

$$P_t = \frac{q-c}{2\beta} \quad \text{(no deviation)} \qquad P_t = c \quad \text{(deviation)}$$

$$\pi^T = \left(\frac{\alpha - c}{2\beta} - c\right) Q$$

$$P = \alpha - \beta Q \quad Q = \frac{\alpha}{P} = \frac{\alpha}{\beta} - \frac{1}{\beta} \cdot \frac{q-c}{2\beta} = \frac{2\alpha\beta - \alpha + c}{2\beta^2}$$

$$\pi^T = \left(\frac{\alpha - c}{2\beta} - c\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right)$$

$$\pi^T = \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right) \cdot \frac{1}{2}$$

$$\pi^D = \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right) \quad \pi^N = 0$$

$$PDV_T = \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right)\left(\frac{1}{2} + \frac{1}{2}\delta + \frac{1}{2}\delta^2 + \cdots\right)$$

$$PDV_T = \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right)\left(\frac{1}{2} \cdot \frac{1}{1-\delta}\right)$$

$$PDV_D = \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right)$$

$$PDV_T \geq PDV_D$$

$$\left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right) \cdot \frac{1}{2} \cdot \frac{1}{1-\delta} \geq \left(\frac{\alpha - c - 2\beta c}{2\beta}\right)\left(\frac{2\alpha\beta - \alpha + c}{2\beta^2}\right)$$

$$\frac{1}{1-\delta} \geq 2 \quad 1 \geq 2 - 2\delta \quad 2\delta \geq 1 \quad \delta \geq \frac{1}{2}$$

In this analysis, we compare a firm's profits under two strategies: cooperating by sticking to a collusive agreement (no deviation) or undercutting the competitor for short-term gain (deviation). The firm's profit is calculated using the standard formula $\pi = (P - c)Q$, where P is price, c is cost, and Q is quantity. Using the demand curve $P = q - \beta Q$, we solve for Q under the collusive price $P = \frac{q-c}{2\beta}$, which gives us the quantity demanded when both firms cooperate. This leads to a profit formula for the no-deviation case, π^T, which is then simplified. For the deviation case, the firm sets price equal to cost $P = c$ to capture the entire market, earning a one-time profit π^D. After that, it earns zero due to a price war. To evaluate which strategy is better in the long run, we calculate the present discounted value (PDV) of profits under each scenario. The PDV of cooperation involves an infinite sum discounted by factor δ, while the PDV of deviation is just π^D. We then set $PDV_T \geq PDV_D$, meaning the present discounted value of cooperating must be at least as large as the value of deviating. This inequality captures the firm's incentive to stick with collusion. By substituting the expressions for PDV_T and PDV_D into the inequality and simplifying, we isolate the discount factor δ, which tells us the minimum level of future value the firm must place on ongoing cooperation for it to be more profitable than a one-time deviation.

2 point for correct setup and calculation of deviated vs. non-deviated profits. 3 points for correct setup of $PDV_T PDV_D$, including the discounting. 2 points for arriving at $\delta = 1/2$.

Part K *(5 Points)* Now assume price can be any value between $(\alpha + c)/2$ and α. Calculate a formula for $\delta(p^*)$, which is a function for the minimum discount rate in order for there to exist a Nash equilibrium where both firms charge a price of p^*.

$$P_t = P^* \quad \text{(no deviation)} \qquad P_t = \alpha \quad \text{(deviation)}$$

$$\pi_T = (P^* - c)Q \quad Q = \frac{\alpha - P^*}{\beta}$$

$$\pi_T = \frac{(P^* - c)(\alpha - P^*)}{2\beta} \qquad \pi_D = \frac{(P^M - c)(\alpha - P^M)}{\beta} = \frac{(\alpha - c)^2}{4\beta} \qquad \pi_N = 0$$

$$PDV_T \geq PDV_D$$

$$\frac{(P^* - c)(\alpha - P^*)}{2\beta} \cdot \frac{1}{1 - \delta} \geq \frac{(\alpha - c)^2}{4\beta}$$

$$(P^* - c)(\alpha - P^*) \cdot \frac{1}{1 - \delta} \geq \frac{(\alpha - c)^2}{2}$$

$$\frac{1}{1 - \delta} \geq \frac{(\alpha - c)^2}{2(P^* - c)(\alpha - P^*)}$$

$$1 - \delta \leq \frac{2(P^* - c)(\alpha - P^*)}{(\alpha - c)^2}$$

$$\delta(P^*) \geq 1 - \frac{2(P^* - c)(\alpha - P^*)}{(\alpha - c)^2}$$

$$\delta(P^*) = 1 - \frac{2(P^* - c)(\alpha - P^*)}{(\alpha - c)^2}$$

In this analysis, we consider a firm's decision to maintain a collusive price P^* or deviate and trigger detection. If the firm has not deviated, it continues charging P^* and earns profit over time. If it has deviated, the market reverts to a less profitable outcome where price equals α. The firm's profit is calculated using the standard formula $\pi = (P - c)Q$, where P is price, c is cost, and $Q = \frac{\alpha - P}{\beta}$ is the resulting quantity from the demand curve. Using this, we find the profit from cooperating at price P^* to be $\pi_T = \frac{(P^* - c)(\alpha - P^*)}{\beta}$. The profit from deviation, π_D, is the one-time gain from undercutting and is given by $\pi_D = \frac{(\alpha - c)^2}{4\beta}$. If the firm deviates, it is punished with zero profit from then on, so $\pi_N = 0$. To determine whether the firm prefers to stick to collusion, we compare the present discounted value (PDV) of cooperation and deviation.

The PDV of cooperation is a geometric series since profits continue indefinitely and are discounted by factor δ, while the PDV of deviation is just the one-time gain. We set $PDV_T \geq PDV_D$ and solve for the discount factor δ to find the condition under which the firm finds it worthwhile to continue cooperating at price P^*.

2 point for correct setup and calculation of deviated vs. non-deviated profits. 3 points for correct formula for $\delta(p*)$.

Problem 6: Two-Period Extraction (25 points)

Please note that this question was released with an error: the plus sign on the discounted term in the first equation below, $\pi(y_0, p_0, p_1)$, was mistakenly written as a minus sign. This slightly changed the interpretation of some of the following questions. However, we graded Question 6 such that **the error did not impact students' scores**. Both the "minus" and "plus" versions of the profit equation were treated as valid for all parts of the question, so no team lost points due to the incorrect formula. We apologize for any confusion and appreciate your understanding.
In this problem, we build out an economic model of natural resource extraction across two time periods.

The utilization of nonrenewable natural resources is a fundamentally dynamic problem: it is crucial to optimize not just how much of a resource is used but also when that use occurs. This problem uses the example of mineral extraction, a live issue in economics and geopolitics today, as window dressing to explore optimal resource consumption.

Part A *(13 points total)* Suppose a firm owns a fixed stock (x) of minerals across its mines. The firm seeks to maximize profits across two periods (t_0 and t_1) by selling some amount of minerals (y_0) in the first and the rest in the second. The firm faces constant marginal costs (c) and discount factor ρ. [Note: ρ is derived from the firm's discount rate and serves to define t_1 profits in present value terms.]

The firm's profits from mineral extraction during both periods in terms of y_0, first period price (p_0), and second period price (p_1) can therefore be expressed as:

$$\pi(y_0, p_0, p_1) = (p_0 - c)y_0 + \rho(p_1 - c)(x - y_0)$$

(1) *(3 points)* Assuming a positive quantity of minerals is sold during each period, the firm will set quantities such that an equilibrium between t_0 and t_1 is reached. This is known as the no-intertemporal arbitrage condition. Derive the first order condition with respect to y_0 of the above equation to find that relationship. Then, provide some brief intuition as to what the condition means for the firm. [Hint: you should find some relationship between p_0 and p_1.] Take the First Order Condition (derive with respect to y_0 and equate to 0): $\frac{d}{dy_0}\pi = 0$, so $(p_0 - c) = \rho(p_1 - c)$. Intuition: the resulting equation represents an equilibrium for which the firm cannot increase profits by shifting extraction between the two periods. **2 points for FOC, 1 point for intuition**

(2) *(5 points)* Given stock size, costs, and mineral demands in the two periods as well as ρ, we are able to predict firm behavior in equilibrium. Solve for mineral prices (p_0 and p_1) and extraction levels (y_0 and y_1) to the nearest hundredth in both periods using the following parameters. How do prices and quantities differ between the periods? [Hint: remember that the entire resource stock is extracted.]

$$y_0 = 16 - p_0$$

$$y_1 = 12 - p_1$$

$$x_0 = 8$$

$$\rho = 0.84$$

$$c = 1.5$$

Plug inverse demands, costs, and rho into the above equation to solve for equilibrium: $16 - y_0 - 1.5 = 0.84(12 - y_1 - 1.5)$. Then, substitute $y_0 + y_1 = x$ to find y_0. In this case,

$$16 - y_0 - 1.5 = 0.84 * (12 - (8 - y_0) - 1.5)$$

so $y_0 = 6.74$. Use y_0 to find y_1, p_0, and p_1. $y_1 = x_0 - y_0 = 1.26$, $p_0 = 9.26$, and $p_1 = 10.74$. Note that more extraction takes place in the near period 0, and prices increase in period 1: $y_0 > y_1$, $p_0 < p_1$ **1 point for setup, 1 point for substitution, 2 points for y and ps, 1 point for difference**

(3) *(3 points)* We might worry about the negative environmental and social impacts of mineral extraction. Suppose a consortium of major companies announce their plans to reduce mineral purchases in period 1, reducing demand to the equation below. All other parameters remain unchanged. Re-calculate the quantities (p_0, p_1, y_0, y_1) from the last task. What happens to extraction and prices in each period?

$$y_1 = 10 - p_1$$

Use the same setup as last time with the new demand equation. $16 - y_0 - 1.5 = 0.84(10 - y_1 - 1.5) = 0.84(10 - (8 - y_0) - 1.5)$. Then solve for y_0, y_1, p_0, and p_1. $y_0 = 7.65$, $y_1 = 0.35$, $p_0 = 8.35$, and $p_1 = 9.65$. Decreased demand in period 1 causes y_0 to increase and y_1 to decrease relative to the previous scenario. Both prices fall. **1 point for setup, 1 point for demand trend, 1 point for price trend**

(4) *(2 points)* German economist Hans-Werner Sinn identifies this phenomenon as the "Green Paradox," where well-intentioned environmental commitments for the future can increase extraction—and its associated harms—today. Explain why setting green demand-side restrictions for the future might still be worthwhile. Then, describe a sustainable policy that Sinn might recommend instead. The "Green Paradox" might not hold when short-run mineral supply is inelastic (due to extraction and/or exploration constraints). Future demand-side restrictions might drive green R&D investment/industrial policy, helping short-run inelastic green firms to scale. They might also increase political palatability. For alternatives: Sinn suggests supply-side fixes (in the carbon context: a global emissions trading system, paying for destruction of carbon fuels, harm mitigation practices like carbon capture and storage that make future extraction more attractive, etc.). **1 point for a reasonable explanation, 1 point for a reasonable policy**

Part B *(12 points total)* In reality, it is unlikely that costs remain static with respect to resource stock and extraction level. We might instead expect prices to rise along both dimensions, with firms utilizing the most accessible resources first.

$$c(x, y) = C(\sigma + x)^{-\alpha} y^{1+\beta}$$

(1) *(4 points)* The above equation represents one way that mineral stocks and extraction rates could influence costs. What do the alpha (α), beta (β), and sigma (σ) parameters indicate about the firm's costs? Given the full-extraction assumption, what must be true about sigma in cases like ours? [Hint: consider cases where the parameters are zero and positive.] β: the marginal cost of extraction (with respect to y) is increasing if β is positive and constant if β is zero. α: costs

increase with lower stocks if α is positive but are constant if α is zero. σ: if zero (and α is positive), costs move toward infinity as stock decreases. The firm will therefore not physically exhaust the resource. Since we assume total exhaustion, σ must be positive in our case. **1 point for α, 1 point for β, 1 point for basic σ intuition, 1 point for σ in our case**

(2) *(2 points)* We can use this advanced cost equation to inform a new equilibrium condition. We continue to adhere to the assumptions that extraction is positive in both periods and that all available minerals are extracted. Use the above relationship to construct a profit equation, similar to the one stated in Part A, in terms of y_0 and x_0. Represent period costs as a function of x and y as shown above. Use a similar setup to the profit equation in A.a, including a term for each period: $\pi(y_0) = p_0 y_0 - c(x_0, y_0) + \rho(p_1(x_0 - y_0) - c(x_1, y_1)$. Ensure that cost functions utilize the correct period terms (i.e., x_1 and y_1 for period 1 cost). **2 point for correct equation**

(3) *(6 points)* To find a mineral firm's profits with variable costs, take the first order condition of the equation you found above with respect to y_0. Arrange the resulting equation such that the discounted terms fall on the same side. Then, briefly describe the real-world significance of each term (i.e., how each term explains changes in profit across some dimension.) [Hint: Take partial derivatives of the cost function. Use your knowledge of the relationship between x and y to determine dx_1/dy_0 and dy_1/dy_0.] The first order condition returns:

$$\frac{d\pi(y_0)}{dy_0} = (p_0 - \frac{dc(x_0, y_0)}{dy_0}) + \rho(p_1 \frac{dy_1}{dy_0} - \frac{dc(x_1, y_1)}{dy_1}\frac{dy_1}{dy_0} - \frac{dc(x_1, y_1)}{dx_1}\frac{dx_1}{dy_0}) = 0$$

Since the firm extracts the entire mineral stock, $x_1 = y_1 = x_0 - y_0$; this implies that $\frac{dy_1}{dy_0} = -1$ and $\frac{dx_1}{dy_0} = -1$. Use that relationship to simplify the FOC to:

$$p_0 - \frac{dc(x_0, y_0)}{dy_0} = \rho(p_1 - \frac{dc(x_1, y_1)}{dy_1} - \frac{dc(x_1, y_1)}{dx_1})$$

The left side term represents higher profit in period 1 due to increased extraction: period 0 marginal profit equals period 0 marginal revenue (price p_0) minus period 0 marginal cost ($\frac{dc(x_0,y_0)}{dy_0}$). The positive discounted term (beginning with p_1) indicates decreasing period 1 profit due to higher period 0 sales—since all of the stock is consumed, sales in period 0 necessarily decrease sales in period 1. The negative discounted term corresponds to reduced profit due to higher stock-dependent costs. Sales in period 0 reduce the period 1 stock (x_1), increasing period 1 costs according to the cost function. **2 points for correct FOC setup, 1 point for $\frac{dy_1}{dy_0}$ and $\frac{dx_1}{dy_0}$ substitutions, 1 point for first term intuition, 1 point for second term, 1 point for third term**